Vertex Transversals that Dominate

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ABSTRACT

For any graph, there is a largest integer k such that given any partition of the vertex set with at most k elements in each class of the partition, there is transversal of the partition that is a dominating set in the graph. Some basic results about this parameter, the *partition domination number,* are obtained. In particular, it is shown that its value is 2 for the two-dimensional infinite grid, and that determining whether a given vertex partition admits a dominating transversal is *NP*-complete, even for a graph which is a simple path. The existence of various dominating transversals in certain partitions in regular graphs is studied as well. © 1996 John Wiley & Sons, Inc.

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1. INTRODUCTION

In this paper we explore a graph-theoretic parameter defined by a statement of the form: "For every partition of the vertex set satisfying P there is a transversal satisfying Q."

Here, and in what follows, a *transversal* of a partition is a set of distinct representatives of the classes of the partition. One example of such a parameter is the *strong partition independence* or *strong chromatic number* of a graph $G = (V, E)$, $pi^*(G)$, studied in [1], [2], [9], and [10]. This is defined to be the least positive integer k such that every partition of V together with an arbitrary number of additional isolated vertices having exactly k vertices in each class admits a partition whose classes are transversals of the original partition as well as independent sets in G. Bounds on this and related parameters for regular graphs are developed by probabilistic methods in [9] and [2]. A conjecture of Du, Hsu, and Hwang [7], modified by Erdös, has recently been settled in the affirmative in [10], where the authors apply the main result of [4] to deduce that the strong chromatic number of the infinite path is 3. Determining the value of this parameter can be difficult even for relatively simple graphs.

In this paper we study the following analogously defined parameter.

Definition. The *partition domination* number $pd(G)$ of a graph $G = (V, E)$ is the largest positive integer k such that every partition of V having at most k vertices in each class admits a transversal that is a dominating set in G.

In [9] it is observed that the partition domination number of the infinite path (or 1-dimensional grid) is 2. In this paper we obtain a number of basic results concerning dominating transversals of vertex partitions in graphs. One of our results shows that the partition domination number of the infinite 2-dimensional grid is also 2. It is worth noting that the strong partition independence number of the 2-dimensional grid is unknown but is between 5 and 9. The lower bound is deduced from the easy inequality $pi^*(G) \geq \Delta(G) + 1$ mentioned in [2], where $\Delta(G)$ is the maximum degree of G, and the upper bound can be seen by applying the easily established inequality $pi^*(G \times H) \leq pi^*(G)pi^*(H)$ to the case where G and H are both infinite paths. Here the product $G \times H$ is the graph whose vertices are all pairs of vertices (u, v), where u is a vertex of G and v is a vertex of H, in which two pairs are adjacent if and only if they are equal in one coordinate and adjacent in the other. Thus when G and H are infinite paths, their product is the infinite 2-dimensional grid.

We feel that these parameters deserve to be studied, having an inviting Ramsey theoretic flavor. Possible applications of the partition independence parameter to fault-tolerant data storage are speculated in [9]. In a similar spirit we describe a fanciful illustration of partition domination in Section 2.

The rest of the paper is organized as follows. In Section 2 we discuss notation and definitions and present some simple observations. In Section 3 we prove that the partition domination number of the 2-dimensional grid is 2. In Section 4 we employ the Lovaśz Local Lemma to prove a bound on the degree of regular graphs which ensures the existence of various dominating transversals of vertex partitions that have classes of uniform size. In Section 5 we consider the complexity of computing a dominating transversal and show that the corresponding decision problem is NP-complete, even for graphs which are simply paths. In Section 6 we conclude with a brief discussion of open problems.

2. PRELIMINARIES

All graphs $G = (V, E)$ in this paper are simple, without loops or multiple edges. An *independent set* of vertices in a graph is a set $V' \subseteq V$ such that for all $x, y \in V'$, $xy \notin E$. A set of vertices $V' \subseteq V$ is a *dominating set* if for every vertex $u \in V$ there is a vertex $w \in V'$ such that $u = w$ or $uw \in E$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G.

By the *d-dimensional grid*, denoted L^d, we refer to the graph having as vertices the integer lattice points in the d-dimensional Euclidean space, with two vertices adjacent if and only if they are at distance 1. By L_n^d we denote the (finite) d-dimensional grid for which the lattice points have coordinates in the range $1, \ldots, n$. Other standard graph-theoretic terminology may be found in [5].

Definition. A partition π of a set X is *k-thick* if for each class $[x]$ of π, $|[x]| \geq k$. Similarly a partition is *k-thin* (*k-exact*) if each class of the partition has at most (exactly, respectively) k elements.

For convenience, we may refer to a partition of the vertex set of a graph as a *coloring*, and refer to the classes of the partition as *color classes*.

Definition. A partition π of a set X is *orthogonal* to a partition $\pi^{\perp}$ of X if each class of π is a transversal of $\pi^{\perp}$ (and vice versa).

Definition. The *partition domination number pd*(*G*) of a graph $G = (V, E)$ is the greatest integer k such that for every k-thin partition π of V there is a transversal T of π that is a dominating set of vertices in G.

The following is a fanciful illustration of how an application of this parameter might arise in a certain distributed or parallel model of computation.

The Sound of the Perfect Chime

A certain society (modeled by a graph G) wishes that each member should be able to hear the Sound of the Perfect Chime. The Sound of the Perfect Chime is produced by an elaborate instrument that can be constructed only with some difficulty. The effort of constructing one can be shared, however. If a group of people work together to construct a chime to produce the Perfect Sound, then they will select a Keeper of the Chime who will have the instrument at her house (i.e., vertex).

Cooperation is made difficult by the fact that many individuals in the society find it impossible to work together for complicated and inscrutable reasons. Yet cooperative groups for building the chimes do emerge.

Once constructed (and located at a particular vertex v), the Sound of the Chime can be heard at v and at all its neighbors.

If we know that $pd(G)$ is k, then we know that if the society forms into cooperative groups to build the instruments, with everyone belonging to one of these groups, and if no group has more than k members, then Keepers of the Chimes can be selected so that everyone will be able to hear the Sound of the Perfect Chime.

We next prove a simple inequality relating to the partition domination number to a graph to its domination number.

Lemma 1. Let G be a graph on n vertices, and let $\gamma(G)$ be the domination number of G. If $\gamma(G) \geq 2$, then $pd(G) < n/[\gamma(G) - 1]$.

Proof. Let $\gamma = \gamma(G) \geq 2$ and let π be any partition of $V(G)$ that has $\gamma - 2$ classes of size $\lceil n/(\gamma - 1) \rceil$ each and one class of the remaining vertices (at most $n/(\gamma - 1)$ remaining). If $k \geq n/(\gamma - 1)$, then π is a k-thin partition of G with only $\gamma - 1$ classes. Therefore π has no dominating transversal. ∎

Corollary 1. If $\gamma(G) \geq 2$, then $pd(G) < [n(\Delta + 1)]/[n - \Delta - 1]$, where Δ is the maximum degree of G.

Proof. Apply Lemma 1 with the simple bound $\gamma(G) \geq n/(\Delta + 1)$. ∎

Corollary 2. If $n \geq (\Delta + 1)(\Delta + 2)$, then $pd(G) \leq \Delta + 1$.

3. THE TWO-DIMENSIONAL GRID

The goal of this section is to determine $pd(L^2)$. We initially consider the finite 2-dimensional square grids L_n^2 of order n^2. By Corollary 2, $pd(L_n^d) \leq 2d + 1$ (except for a few small values of n).

Consider the following partial 3-thin partition of L_n^2 where the elements of each class are represented by indexed letters and where the classes containing the vertices labeled A, B, C, D are not determined.

$$\begin{array}{cccccc}
 & B & & & & \\
A & f_1 & C & & & \\
 & D & g_1 & f_2 & & \\
 & & f_3 & h_1 & g_2 & \\
 & k_3 & l_2 & g_3 & i_1 & h_2 \\
k_2 & l_1 & i_3 & j_1 & h_3 & \\
 & j_3 & k_1 & i_2 & & \\
 & & j_2 & & &
\end{array}$$

We call such a configuration a *j-shape* $(A, B, C, D, f, g, \ldots, l)$ (since it looks like a slanted j when l_2 is removed).

Lemma 2. Let T be a transversal of any completion to a 3-thin partition of the vertex set of L_n^2, of a 3-thin partition of a *j-shape* $(A, B, C, D, f, g, \ldots, l)$ for $n \geq 8$. If T dominates L_n^2, then at least one of the vertices labeled A, B, C, D is in T.

Proof. Suppose, to the contrary, that A, B, C, and D are not in T. Since T dominates L_n^2, $f_1 \in T$. Thus f_2 and f_3 are not in T (T is a transversal) and since C and D are not in T, $g_1 \in T$. This forces h_1 to be in T, which in turn forces i_1 to be in T, and so on, so that j_1, k_1, and l_1 are all in T. Thus f_3, g_3, i_3, and k_3 are not in T. But then l_2 is not dominated by T, a contradiction.

Theorem 1. For all $n \geq 130$, $pd(L_n^2) = 2$.

Proof. To prove that $pd(L_n^2) \leq 2$, a partial 3-thin partition is constructed so that any completion to a 3-thin partition of the entire graph fails to have a dominating transversal.

Overlap a *j-shape* $(a_2, a_3, b_2, b_3, f^1, g^1, \ldots, l^1)$, denoted J_1 with another *j-shape* denoted J_2, *j-shape* $(a_3, a_2, c_2, c_3, f^2, g^2, \ldots, l^2)$, that is rotated by 180 degrees, so that they intersect in the following way:

$$\begin{array}{cccccc}
h_1^2 & f_3^2 & & & & \\
f_2^2 & g_1^2 & c_3 & & & \\
 & c_2 & f_1^2 & a_3 & & \\
 & & a_2 & f_1^1 & b_2 & \\
 & & & b_3 & g_1^1 & f_2^1 \\
 & & & & f_3^1 & h_1^1
\end{array}$$

(Note that f^1 is a class of the partition containing the vertices labeled f_1^1, f_2^1, f_3^1, and it is different from the class f^2.) Overlap another *j-shape* $(d_2, b_1, c_1, d_3, f^3, g^3, \ldots, l^3)$, J_3, with a *j-shape* $(d_1, b_1, e_2, e_3, f^4, g^4, \ldots, l^4)$, J_4 that is rotated by 180 degrees, and with an additional *j-shape* $(c_1, b_1, d_1, e_1, f^5, g^5, \ldots, l^5)$ which is rotated by 270 degrees, so that they intersect in the following way:

$$\begin{array}{cccccc}
h_1^4 & f_3^4 & & & & \\
f_2^4 & g_1^4 & e_3 & & f_2^5 & h_1^5 \\
 & e_2 & f_1^4 & d_1 & g_1^5 & f_3^5 \\
 & & b_1 & f_1^5 & e_1 & \\
 & d_2 & f_1^3 & c_1 & & \\
 & & d_3 & g_1^3 & f_2^3 & \\
 & & & f_3^3 & h_1^3 &
\end{array}$$

Let this partial 3-thin partition be called J^*. Complete J^* to a 3-thin partition that uses the label a_1 on a vertex somewhere else in the graph. Suppose T is a dominating transversal of the partition and suppose $a_1 \in T$.

Since a_2 and a_3 are not in T, using J_1 and Lemma 2 we have that one of b_2 or b_3 is in T, and hence $b_1 \notin T$. Similarly, using J_2, $c_1 \notin T$. By Lemma 2 again using J_3, one of d_2 or d_3 is in T since b_1 and c_1 are not in T. Hence $d_1 \notin T$, and by Lemma 2 once more using J_4, $e_1 \notin T$. But this is a contradiction, since by Lemma 2 using J_5, one of b_1, c_1, d_1, or e_1 is in T. Therefore, $a_1 \notin T$.

Finally, partition 5 disjoint subgraphs of L_n^2 in the same manner as J^* but with each copy of the partial partition being distinct so as to obtain a partial 3-thin partition. Let $a_1^1, a_1^2, \ldots, a_1^5$ be the unused labels corresponding to a_1. Somewhere else in the graph label the closed neighborhood of a vertex in the following way:

$$\begin{array}{ccc}
 & a_1^1 & \\
a_1^4 & a_1^5 & a_1^2 \\
 & a_1^3 &
\end{array}$$

Complete this partial 3-thin partition to a 3-thin partition for the entire graph. If T is any dominating transversal of the partition, then a_1^i is not in T for $i = 1, 2, \ldots, 5$, by the above argument. Thus a_1^5 is not dominated by T and hence no such transversal exists. Therefore $pd(L_n^2) \leq 2$.

To see that $pd(L_n^2) \geq 2$, let π be a 2-thin partition of L_n^2 and let $Q_1, Q_2, \ldots, Q_t$ be vertex disjoint paths of length 0 or 1 of L_n^2 such that every vertex of L_n^2 is on one of the paths and at most one Q_i has length 0. Define a bipartite graph $H = (X, \pi)$, where $X = \{Q_1, Q_2, \ldots, Q_t\}$ and for all $[x] \in \pi$ and $1 \leq i \leq t$, $Q_i[x] \in E(H)$ if and only if $Q_i \cap [x] \neq \emptyset$. This graph has a matching M which saturates X and hence we can select vertices v_i such that $v_i \in Q_i \cap [x_i]$, where $Q_i \cap [x_i] \in M$. Since the set $T = \{v_1, v_2, \ldots, v_t\}$ dominates L_n^2 and M is a matching, we may add any other vertices of L_n^2 to T so that it becomes a transversal of π. This transversal will also dominate L_n^2 and so $pd(L_n^2) \geq 2$. ▮

Since the same argument shows that $pd(L^2) \geq 2$ we obtain the following.

Corollary 3. $pd(L^2) = 2$.

4. BOUNDS FOR REGULAR GRAPHS

In this section we study bounds on the degree of regular graphs that insure the existence of dominating transversals for certain kinds of partitions. The following has been previously established.

Theorem 2. [9]. If G is a δ-regular graph with $\delta \geq 18\, k \log_2 k$, $k \geq 2$, then $pd(G) \geq k$. ▮

One might wonder whether the regularity condition in the above theorem might be replaced by a *minimum degree* requirement. The Lemma below shows that any generalization of Theorem 2 in this direction fails.

Lemma 3. For all δ, there exists a graph G with minimum degree δ such that $pd(G) = 1$.

Proof. Let δ be given. The graph G that we describe is bipartite on the classes of vertices V_0, V_1, with $|V_0| = 2^{\delta+1}$ and $|V_1| = 2\delta$. We also describe a 2-exact partition π of $V(G)$ that has no dominating transversal.

Let the vertices of V_1 be partitioned into δ pairs. Each pair is a class of π. Let S be the set of all 2^δ distinct subsets of V_1 that contain exactly one vertex from each of the δ pairs. For each $X \in S$, there are two vertices x_0 and x_1 of V_0 which are adjacent to all members of X (and only to them).

Each such pair x_0, x_1 forms a class of π. Thus π is 2-exact. Let T be any transversal of π. Then for some $Y \in S$, $Y \cap T = \emptyset$. Let y_0, y_1 be the two vertices in V_0 whose set of neighbors is Y. Since either y_0 or y_1 is not in T, and since both of them have no neighbor in T, we conclude that T is not a dominating set in G. ▮

Our next result gives a generalization of Theorem 2 that parallels similar results established in [9] and [2] for independent transversals.

Theorem 3. Let $G = (V, E)$ be a finite, δ-regular graph and let π be a k-exact partition of V. If

$$\delta \geq k \log_e(\delta^2 k^2) + k$$

then there exists a partition $\pi^\perp$ of V orthogonal to π with each class of $\pi^\perp$ a dominating set in G. Note that the last inequality holds for $\delta \geq (4 + o(1))k \log_e k$, where the $o(1)$ term tends to zero as k tends to infinity.

The proof of the theorem is probabilistic and is based on the Lovász Local Lemma (cf., e.g., [3], [11]). Let $A_1, \ldots, A_n$ be events in a probability space. A graph H on the vertices $\{1, \ldots, n\}$ (the indices for the A_i) is called a *dependency graph* for $A_1, \ldots, A_n$ if for all i the event A_i is mutually independent of all A_j with $\{i, j\} \notin H$.

The Local Lemma. Assume that for all i, $\Pr(A_i) \leq p$ and let d be the maximum degree of a vertex in H. If $e(d + 1)p \leq 1$ then

$$\Pr\Big(\bigcap_{i=1}^{n} \overline{A}_i\Big) > 0 .$$

Proof of Theorem 3. For each class $[x]$ of π, randomly and independently, choose a permutation of the k vertices of $[x]$ according to a uniform distribution. For each vertex v of G, let E_v denote the event that for some i, $1 \leq i \leq k$, no neighbor u of v is the ith element of the permutation chosen for $[u]$. Note that if the permutations can be chosen so that no event E_v holds, then the sets D_i consisting of the ith vertex in each class of π constitute an orthogonal partition $\pi^{\perp}$ with the required properties.

We first show that for each $v \in V$:

$$\Pr(E_v) \leq k\Big(\frac{k-1}{k}\Big)^{\delta} .$$

To see this, fix an $i \in \{1, \ldots, k\}$, and let us estimate the probability that v will have no neighbor chosen as the ith element in its class. If a class contains j neighbors of v, the probability that the ith element in the randomly chosen permutation of the members of this class is not a neighbor of v is $1 - j/k \leq (1 - 1/k)^j$. Therefore, if π partitions the neighbors of v into nonempty classes of sizes $j_1, \ldots, j_s$ it follows that the probability that no neighbor will be chosen as the ith element in its class is at most

$$\prod_{q=1}^{s} (1 - 1/k)^{j_q} = \Big(\frac{k-1}{k}\Big)^{\delta} .$$

As there are k possible values of i, the desired upper bound for $\Pr(E_v)$ follows.

We now define a dependency graph H for the events E_v as follows. The set of vertices of H is the set of all vertices v of G. Two vertices w and v are adjacent in H if and only if there is a class of the partition π that contains a neighbor of w and a neighbor of v. It is not difficult to check that this is indeed a dependency graph for the events E_v. In fact, if v is a vertex, even if we know the chosen permutations for all the classes of the partition π besides those that contain neighbors of v, the probability of E_v remains unchanged, and it is easy to see that any assumption on the events E_w for vertices w which are not H-neighbors of v is determined by these permutations. It thus follows that the maximum degree of a vertex in H is less than $\delta^2 k$. Therefore, by the Local Lemma, if

$$ek\delta^2 k(1 - 1/k)^{\delta} \leq 1 ,$$

then with positive probability no event E_v holds. It is therefore enough to assume that

$$ek^2\delta^2 e^{-\delta/k} \leq 1 ,$$

implying the assertion of the theorem. ∎

The $\Theta(k \log k)$ estimate in Theorem 3 is tight, up to a constant factor. This is because there are d-regular graphs on n vertices for which the usual domination number is $\Theta((n \log d)/d)$. There are several known examples of such graphs. The best known is, perhaps, the Paley graphs (they give the existence of the required graphs for an infinite dense set of values of d from which it is easy to deduce the existence for every d). See, for example [6, p. 319]. Random d-regular graphs also have the required property, as mentioned, e.g., in [3, p. 6–7]. Applying Lemma 1 to these graphs, it follows that there are d-regular graphs G for which $pd(G) \leq O(d/\log d)$ and hence Theorem 3 (as well as Theorem 2) is tight, up to a constant factor.

5. COMPLEXITY ISSUES

In this section we consider the computational complexity of deciding whether a vertex partition of a graph admits a dominating transversal. Not surprisingly, the problem is NP-complete. The interesting thing is that the problem is NP-complete even for graphs which are simply paths.

Dominating Transversal
Instance: A vertex coloring γ of a graph $G = (V, E)$.
Question: Is there a transversal T of γ that is a dominating set of vertices in G?

Theorem 4. Dominating Transversal is NP-complete, even when restricted to inputs for which G is a path.

Proof. The problem is clearly in NP, since given a transversal T of a coloring γ we can easily check in polynomial time whether T is a dominating set of vertices. To show that the problem is NP-hard we reduce from 3SAT.

Given a Boolean expression E, we will produce a coloring γ_E of a path P with the property that γ_E has a dominating transversal if and only if E is satisfiable, and γ_E will be produced in time polynomial in $|E|$.

It is convenient to introduce the following notation. By a coloring ρ of P_n, the path with n vertices, we mean a function $\rho : [n] \to C$, where $[i] = \{1, 2, \ldots, i\}$ and $C \subseteq \mathbf{Z}$. We regard the vertex set of P_n to be the integers $[n]$. To avoid confusion, integers representing colors are printed in bold. We may also write $\rho = (x_1, x_2, \ldots, x_n)$ to indicate that $\rho(i) = x_i$, and we term this the *sequence representation* of ρ. Given ρ_1 and ρ_2, two distinct colorings of paths, $\rho_1 = (x_1, \ldots, x_n)$ and $\rho_2 = (y_1, \ldots, y_m)$, we define $\rho_1 * \rho_2$ to be the coloring of P_{n+m} with the sequence representation $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

The coloring γ_E is defined as a product, in terms of the above operation, of various *component* colorings:

$$\gamma_E = \alpha_0 * \prod_{v \in \mathrm{var}(E)} \alpha_v * \prod_{C \in \mathrm{clauses}(E)} \beta_C \tag{1}$$

where α_0, α_v, and β_C are described in the following paragraphs. One of the components is designed to force a particular vertex of a particular color ($\mathbf{1}$, in what follows) to belong to every dominating transversal of γ_E. We next describe this component.

Let $\alpha_0 = (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{4}, \mathbf{1})$. According to our notational conventions, α_0 is a coloring of P_9 and the vertex set of P_9 is the set of integers $\{1, 2, \ldots, 9\}$. Thus $\alpha_0(4) = \mathbf{2}$ and $\alpha_0(5) = \mathbf{1}$. We argue that $5 \in T$ for any dominating transversal T of α_0 (and therefore of the coloring γ_E which we will construct, that has α_0 as a factor). If $5 \notin T$, then either $4 \in T$ or $6 \in T$. If $4 \in T$, then $6 \notin T$ and $7 \notin T$, and T fails to be a dominating set since the vertex 6 has no neighbor in T. A similar contradiction is reached if we assume $6 \in T$. The colors $\mathbf{2}$, $\mathbf{3}$, and $\mathbf{4}$ of vertices of α_0 will not appear outside of γ_E. Thus, in any dominating transversal T

of α_0 (and γ_E) we must have $2 \in T$ and $8 \in T$ since the **3** and **4** color classes each contain only a single vertex. The purpose of this component is to supply a color **(1)** so that vertices of this color elsewhere in γ_E are forced *not* to belong to any dominating transversal T. For convenience, we will reuse the color names **2, 3, 4,** in describing other components. This should cause no confusion.

Let δ denote the coloring $\delta = (\mathbf{1}, i, \mathbf{1})$ where i is a color used nowhere else in γ_E. We use δ extensively in constructing the variable and clause components of γ_E. Our convention is that for each occurrence of δ in our descriptions, there is a unique color used only once (on the corresponding interior vertex). Let D denote the set of colors thus reserved by the occurrences of δ in γ_E.

For each variable v that occurs in the Boolean expression E we create a component α_v. Associated with v is set of colors $I_v = I_v^+ \cup I_v^-$ with $(\{\mathbf{1}\} \cup D) \cap I_v^+ = \emptyset$, $(\{\mathbf{1}\} \cup D) \cap I_v^- = \emptyset$ and $I_v^+ \cap I_v^- = \emptyset$. If u is a variable of E distinct from v, then $I_u \cap I_v = \emptyset$. Furthermore, we distinguish the colors $i_v^+ \in I_v^+$ and $i_v^- \in I_v^-$ and establish a bijection between I_v^+ and I_v^- which we write additively: $I_v^- = \{-x | x \in I_v^+\}$ with $i_v^+ = -i_v^-$.

The functioning of α_v is most easily explained by a small example. Suppose the variable v occurs (either negated or unnegated) in five clauses. A suitable α_v would be described:

$$\begin{aligned}\alpha_v = &\delta * (\mathbf{-10}, \mathbf{-5}, \mathbf{-9}) * \delta * (\mathbf{-8}, \mathbf{-4}, \mathbf{-7}) * \delta * (\mathbf{-6}, \mathbf{-3}, \mathbf{-5}) * \\ &\delta * (\mathbf{-4}, \mathbf{-2}, \mathbf{-3}) * \delta * (\mathbf{-2}, \mathbf{2}) * \delta * (\mathbf{3}, \mathbf{2}, \mathbf{4}) * \delta * (\mathbf{5}, \mathbf{3}, \mathbf{6}) * \\ &\delta * (\mathbf{7}, \mathbf{4}, \mathbf{8}) * \delta * (\mathbf{9}, \mathbf{5}, \mathbf{10}) * \delta\end{aligned}$$

Note that in the middle of α_v is the sequence $(\mathbf{-2}, \mathbf{2})$. This is the key to the components' functioning. Here we have $i_v^+ = \mathbf{2}$, $i_v^- = \mathbf{-2}$, and $I_v^+ = \{\mathbf{2}, \mathbf{3}, \ldots, \mathbf{10}\}$.

In the argument we assume that α_0 and α_v are factors of γ_E. In particular, no vertex colored **1** in α_v can belong to a dominating transversal T (because a representative of the color class **1** is forced to be chosen in α_0), and hence every internal vertex of an occurrence of δ in α_v must be in T. Our example, is by our notational convention, a coloring of P_{56}, with $\alpha_v(28) = \mathbf{-2}$ and $\alpha_v(29) = \mathbf{2}$. In a dominating transversal T, necessarily either $28 \in T$ or $29 \in T$ (or both). This necessity encodes the "decision" concerning the truth-value of the variable v.

If $28 \in T$, then $23 \notin T$ (the vertex 23 is also colored **–2**), and therefore $22 \in T$ and $24 \in T$. By further "cascading" we may conclude that representatives of the color classes $\mathbf{-2}, \mathbf{-3}, \ldots, \mathbf{-10}$ must all be chosen from the vertices of the component α_v. If $29 \in T$, then the situation is quite symmetric. The pairs of colors $\pm\mathbf{10}$, $\pm\mathbf{9}$, $\pm\mathbf{8}$, $\pm\mathbf{7}$, and $\pm\mathbf{6}$ (corresponding to the five occurrences of literals of v in E presumed for this example) communicate the results of the "decision" to the clause components (described below) in the following way: if $28 \in T$ (symmetrically, if $29 \in T$) then T can dominate α_v with $29 \notin T$ and, by the appropriate choices, with no representatives of the color classes $\mathbf{6}, \mathbf{7}, \ldots, \mathbf{10}$ chosen in α_v. Thus, the representatives of these classes are free to be chosen elsewhere (in particular, they may be chosen from the vertices of the clause components).

The above small example generalizes in a straightforward manner to a construction accommodating any number of clauses in which literals of the variable v might occur. Indeed, the same construction allows us to supply "multiple copies" of any variable component's "decision". In the clause components, we actually employ two copies of each "decision".

For each clause component we reserve two colors j and j' that are used nowhere else. The variable components serve to supply colors to the clause components. According to the "decision" in the variable component, representatives of the supplied colors may be forced to be chosen in the variable component (with the consequence that vertices of these colors in the clause components cannot belong to T) or representatives may be free to be chosen (in a transversal that dominates the variable components) from among the vertices of the clause components.

For an example, suppose the clause C of E is (written additively), $C = (x + \overline{y} + z)$. Abusing our notation, suppose that the variable component α_x supplies the colors x and x', suppose the variable component α_y supplies the colors $-y$ and $-y'$ and suppose the variable component α_z supplies the colors z and z'. By "supplying" colors we mean that, for example, $\{x, x'\} \subseteq I_x^+$ and that in the construction of the variable component α_x, the colors x and x' are "specially created" for the clause C. Then the clause component β_C of γ_E is described by the sequence representation

$$\beta_C = \delta * (j, x, -y, z, x', -y', z', j') * \delta .$$

If T is a transversal that dominates the variable components, then T dominates the vertices of β_C if and only if the representatives of the colors x and x', or of $-y$ and $-y'$, or of z and z' are chosen from the vertices of β_C. This will be possible if at least one of the "decisions" in the variable components is favorable.

By the definition of γ_E in (1), if E is satisfiable, then we can produce a dominating transversal T of γ_E, and if E is unsatisfiable, then for every possible set of "decisions" in the variable components there will be some clause C (such as, for illustration, the example described above) for which representatives of the supplied colors ($\{x, x', -y, -y', z, z'\}$ in the example) are all forced to be chosen from among the vertices of the variable components, and thus some of the internal vertices of β_C fail to be dominated. ∎

6. CONCLUSIONS AND OPEN PROBLEMS

We have shown that $pd(L^1) = pd(L^2) = 2$. For $d \geq 3$ the exact value of $pd(L^d)$ is not known, although it follows from Corollary 2 and Theorem 2 (and its proof together with as standard compactness argument) that

$$\Omega(d/\log d) \leq pd(L^d) \leq 2d + 1$$

for all d. It would be interesting to determine precisely the value of $pd(L^d)$ for additional values of d. More generally, it may be interesting to investigate the relationship between the partition domination number of two graphs and their product. It is not too difficult to prove that $pd(G \times H) < (pd(G) + 1)(\delta(H) + 1)$, where $\delta(H)$ is the minimum degree of H, and there may be deeper similar relations.

The *choice number* of a graph, introduced in [12] and in [8] and studied in numerous subsequent papers extends the notion of the chromatic number of a graph. There is a natural way to define the "choice analog" of the strong chromatic number mentioned in the introduction, and the study of this parameter may be interesting.

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